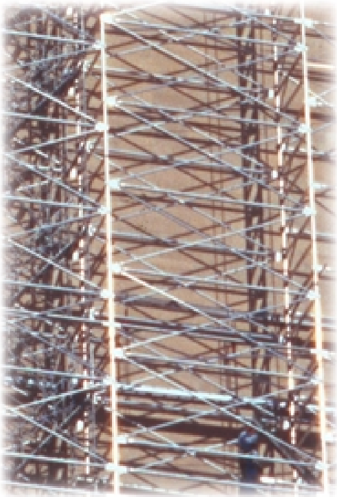

CHAPTER 8

Perron–Frobenius Theory of Nonnegative Matrices

8.1 INTRODUCTION

$\mathbf{A} \in \Re^{m\times n}$ is said to be a ***nonnegative matrix*** whenever each $a_{ij} \geq 0$, and this is denoted by writing $\mathbf{A} \geq \mathbf{0}$. In general, $\mathbf{A} \geq \mathbf{B}$ means that each $a_{ij} \geq b_{ij}$. Similarly, $\mathbf{A}$ is a ***positive matrix*** when each $a_{ij} > 0$, and this is denoted by writing $\mathbf{A} > \mathbf{0}$. More generally, $\mathbf{A} > \mathbf{B}$ means that each $a_{ij} > b_{ij}$.

Applications abound with nonnegative and positive matrices. In fact, many of the applications considered in this text involve nonnegative matrices. For example, the connectivity matrix $\mathbf{C}$ in Example 3.5.2 (p. 100) is nonnegative. The discrete Laplacian $\mathbf{L}$ from Example 7.6.2 (p. 563) leads to a nonnegative matrix because $(4\mathbf{I} - \mathbf{L}) \geq \mathbf{0}$. The matrix $e^{\mathbf{A}t}$ that defines the solution of the system of differential equations in the mixing problem of Example 7.9.7 (p. 610) is nonnegative for all $t \geq 0$. And the system of difference equations $\mathbf{p}(k) = \mathbf{A}\mathbf{p}(k-1)$ resulting from the shell game of Example 7.10.8 (p. 635) has a nonnegative coefficient matrix $\mathbf{A}$.

Since nonnegative matrices are pervasive, it's natural to investigate their properties, and that's the purpose of this chapter. A primary issue concerns the extent to which the properties $\mathbf{A} > \mathbf{0}$ or $\mathbf{A} \geq \mathbf{0}$ translate to spectral properties—e.g., to what extent does $\mathbf{A}$ have positive (or nonnegative) eigenvalues and eigenvectors?

The topic is called the "Perron–Frobenius theory" because it evolved from the contributions of the German mathematicians Oskar (or Oscar) Perron[89] and

[89] Oskar Perron (1880–1975) originally set out to fulfill his father's wishes to be in the family busi-

Ferdinand Georg Frobenius.[90] Perron published his treatment of positive matrices in 1907, and in 1912 Frobenius contributed substantial extensions of Perron's results to cover the case of nonnegative matrices.

In addition to saying something useful, the Perron–Frobenius theory is elegant. It is a testament to the fact that beautiful mathematics eventually tends to be useful, and useful mathematics eventually tends to be beautiful.

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ness, so he only studied mathematics in his spare time. But he was eventually captured by the subject, and, after studying at Berlin, Tübingen, and Göttingen, he completed his doctorate, writing on geometry, at the University of Munich under the direction of Carl von Lindemann (1852–1939) (who first proved that π was transcendental). Upon graduation in 1906, Perron held positions at Munich, Tübingen, and Heidelberg. Perron's career was interrupted in 1915 by World War I in which he earned the Iron Cross. After the war he resumed work at Heidelberg, but in 1922 he returned to Munich to accept a chair in mathematics, a position he occupied for the rest of his career. In addition to his contributions to matrix theory, Perron's work covered a wide range of other topics in algebra, analysis, differential equations, continued fractions, geometry, and number theory. He was a man of extraordinary mental and physical energy. In addition to being able to climb mountains until he was in his midseventies, Perron continued to teach at Munich until he was 80 (although he formally retired at age 71), and he maintained a remarkably energetic research program into his nineties. He published 18 of his 218 papers *after* he was 84.

[90] Ferdinand Georg Frobenius (1849–1917) earned his doctorate under the supervision of Karl Weierstrass (p. 589) at the University of Berlin in 1870. As mentioned earlier, Frobenius was a mentor to and a collaborator with Issai Schur (p. 123), and, in addition to their joint work in group theory, they were among the first to study matrix theory as a discipline unto itself. Frobenius in particular must be considered along with Cayley and Sylvester when thinking of core developers of matrix theory. However, in the beginning, Frobenius's motivation came from Kronecker (p. 597) and Weierstrass, and he seemed oblivious to Cayley's work (p. 80). It was not until 1896 that Frobenius became aware of Cayley's 1857 work, *A Memoir on the Theory of Matrices,* and only then did the terminology "matrix" appear in Frobenius's work. Even though Frobenius was the first to give a rigorous proof of the Cayley–Hamilton theorem (p. 509), he generously attributed it to Cayley in spite of the fact that Cayley had only discussed the result for 2×2 and 3×3 matrices. But credit in this regard is not overly missed because Frobenius's extension of Perron's results are more substantial, and they alone may keep Frobenius's name alive forever.

8.2 POSITIVE MATRICES

The purpose of this section is to focus on matrices $\mathbf{A}_{n\times n} > \mathbf{0}$ with positive entries, and the aim is to investigate the extent to which this positivity is inherited by the eigenvalues and eigenvectors of $\mathbf{A}$.

There are a few elementary observations that will help along the way, so let's begin with them. First, notice that

$$\mathbf{A} > \mathbf{0} \implies \rho(\mathbf{A}) > 0 \tag{8.2.1}$$

because if $\sigma(\mathbf{A}) = \{0\}$, then the Jordan form for $\mathbf{A}$, and hence $\mathbf{A}$ itself, is nilpotent, which is impossible when each $a_{ij} > 0$. This means that our discussions can be limited to positive matrices having spectral radius 1 because $\mathbf{A}$ can always be normalized by its spectral radius—i.e., $\mathbf{A} > \mathbf{0} \iff \mathbf{A}/\rho(\mathbf{A}) > \mathbf{0}$, and $\rho(\mathbf{A}) = r \iff \rho(\mathbf{A}/r) = 1$. Other easily verified observations are

$$\mathbf{P} > \mathbf{0},\ \mathbf{x} \geq \mathbf{0},\ \mathbf{x} \neq \mathbf{0} \implies \mathbf{P}\mathbf{x} > \mathbf{0}, \tag{8.2.2}$$

$$\mathbf{N} \geq \mathbf{0},\ \mathbf{u} \geq \mathbf{v} \geq \mathbf{0} \implies \mathbf{N}\mathbf{u} \geq \mathbf{N}\mathbf{v}, \tag{8.2.3}$$

$$\mathbf{N} \geq \mathbf{0},\ \mathbf{z} > \mathbf{0},\ \mathbf{N}\mathbf{z} = \mathbf{0} \implies \mathbf{N} = \mathbf{0}, \tag{8.2.4}$$

$$\mathbf{N} \geq \mathbf{0},\ \mathbf{N} \neq \mathbf{0},\ \mathbf{u} > \mathbf{v} > \mathbf{0} \implies \mathbf{N}\mathbf{u} > \mathbf{N}\mathbf{v}. \tag{8.2.5}$$

In all that follows, the bar notation $|\star|$ is used to denote a matrix of absolute values—i.e., $|\mathbf{M}|$ is the matrix having entries $|m_{ij}|$. The bar notation will *never* denote a determinant in the sequel. Finally, notice that as a simple consequence of the triangle inequality, it's always true that $|\mathbf{A}\mathbf{x}| \leq |\mathbf{A}|\,|\mathbf{x}|$.

Positive Eigenpair

If $\mathbf{A}_{n\times n} > \mathbf{0}$, then the following statements are true.

- $\rho(\mathbf{A}) \in \sigma(\mathbf{A})$. (8.2.6)
- If $\mathbf{A}\mathbf{x} = \rho(\mathbf{A})\,\mathbf{x}$, then $\mathbf{A}|\mathbf{x}| = \rho(\mathbf{A})\,|\mathbf{x}|$ and $|\mathbf{x}| > \mathbf{0}$. (8.2.7)

In other words, $\mathbf{A}$ has an eigenpair of the form $(\rho(\mathbf{A}), \mathbf{v})$ with $\mathbf{v} > 0$.

Proof. As mentioned earlier, it can be assumed that $\rho(\mathbf{A}) = 1$ without any loss of generality. If $(\lambda, \mathbf{x})$ is any eigenpair for $\mathbf{A}$ such that $|\lambda| = 1$, then

$$|\mathbf{x}| = |\lambda|\,|\mathbf{x}| = |\lambda\mathbf{x}| = |\mathbf{A}\mathbf{x}| \leq |\mathbf{A}|\,|\mathbf{x}| = \mathbf{A}\,|\mathbf{x}| \implies |\mathbf{x}| \leq \mathbf{A}\,|\mathbf{x}|. \tag{8.2.8}$$

The goal is to show that equality holds. For convenience, let $\mathbf{z} = \mathbf{A}\,|\mathbf{x}|$ and $\mathbf{y} = \mathbf{z} - |\mathbf{x}|$, and notice that (8.2.8) implies $\mathbf{y} \geq \mathbf{0}$. Suppose that $\mathbf{y} \neq \mathbf{0}$—i.e., suppose that some $y_i > 0$. In this case, it follows from (8.2.2) that $\mathbf{Ay} > \mathbf{0}$ and $\mathbf{z} > \mathbf{0}$, so there must exist a number $\epsilon > 0$ such that $\mathbf{Ay} > \epsilon\,\mathbf{z}$ or, equivalently,

$$\frac{\mathbf{A}}{1+\epsilon}\mathbf{z} > \mathbf{z}.$$

Writing this inequality as $\mathbf{Bz} > \mathbf{z}$, where $\mathbf{B} = \mathbf{A}/(1+\epsilon)$, and successively multiplying both sides by $\mathbf{B}$ while using (8.2.5) produces

$$\mathbf{B}^2\mathbf{z} > \mathbf{Bz} > \mathbf{z}, \quad \mathbf{B}^3\mathbf{z} > \mathbf{B}^2\mathbf{z} > \mathbf{z}, \quad \ldots \implies \mathbf{B}^k\mathbf{z} > \mathbf{z} \quad \text{for all } k = 1, 2, \ldots.$$

But $\lim_{k\to\infty} \mathbf{B}^k = \mathbf{0}$ because $\rho(\mathbf{B}) = \sigma\big(\mathbf{A}/(1+\epsilon)\big) = 1/(1+\epsilon) < 1$ (recall (7.10.5) on p. 617), so, in the limit, we have $\mathbf{0} > \mathbf{z}$, which contradicts the fact that $\mathbf{z} > \mathbf{0}$. Since the supposition that $\mathbf{y} \neq \mathbf{0}$ led to this contradiction, the supposition must be false and, consequently, $\mathbf{0} = \mathbf{y} = \mathbf{A}|\mathbf{x}| - |\mathbf{x}|$. Thus $|\mathbf{x}|$ is an eigenvector for $\mathbf{A}$ associated with the eigenvalue $1 = \rho(\mathbf{A})$. The proof is completed by observing that $|\mathbf{x}| = \mathbf{A}|\mathbf{x}| = \mathbf{z} > \mathbf{0}$. ∎

Now that it's been established that $\rho(\mathbf{A}) > 0$ is in fact an eigenvalue for $\mathbf{A} > \mathbf{0}$, the next step is to investigate the index of this special eigenvalue.

Index of $\rho(\mathrm{A})$

If $\mathbf{A}_{n\times n} > \mathbf{0}$, then the following statements are true.

- $\rho(\mathbf{A})$ is the only eigenvalue of $\mathbf{A}$ on the spectral circle.
- $index(\rho(\mathbf{A})) = 1$. In other words, $\rho(\mathbf{A})$ is a *semisimple* eigenvalue. Recall Exercise 7.8.4 (p. 596).

Proof. Again, assume without loss of generality that $\rho(\mathbf{A}) = 1$. We know from (8.2.7) on p. 663 that if $(\lambda, \mathbf{x})$ is an eigenpair for $\mathbf{A}$ such that $|\lambda| = 1$, then $\mathbf{0} < |\mathbf{x}| = \mathbf{A}|\mathbf{x}|$, so $0 < |x_k| = \big(\mathbf{A}|\mathbf{x}|\big)_k = \sum_{j=1}^n a_{kj}|x_j|$. But it's also true that $|x_k| = |\lambda|\,|x_k| = |(\lambda\mathbf{x})_k| = |(\mathbf{Ax})_k| = \big|\sum_{j=1}^n a_{kj}x_j\big|$, and thus

$$\Big|\sum_j a_{kj}x_j\Big| = \sum_j a_{kj}|x_j| = \sum_j |a_{kj}x_j|. \tag{8.2.9}$$

For nonzero vectors $\{\mathbf{z}_1, \ldots, \mathbf{z}_n\} \subset \mathcal{C}^n$, it's a fact that $\|\sum_j \mathbf{z}_j\|_2 = \sum_j \|\mathbf{z}_j\|_2$ (equality in the triangle inequality) if and only if each $\mathbf{z}_j = \alpha_j\mathbf{z}_1$ for some $\alpha_j > 0$ (Exercise 5.1.10, p. 277). In particular, this holds for scalars, so (8.2.9) insures the existence of numbers $\alpha_j > 0$ such that

$$a_{kj}x_j = \alpha_j(a_{k1}x_1) \quad \text{or, equivalently,} \quad x_j = \pi_j x_1 \quad \text{with } \pi_j = \frac{\alpha_j a_{k1}}{a_{kj}} > 0.$$

In other words, if $|\lambda| = 1$, then $\mathbf{x} = x_1\mathbf{p}$, where $\mathbf{p} = (1, \pi_2, \ldots, \pi_n)^T > \mathbf{0}$, so

$$\lambda\mathbf{x} = \mathbf{A}\mathbf{x} \implies \lambda\mathbf{p} = \mathbf{A}\mathbf{p} = |\mathbf{A}\mathbf{p}| = |\lambda\mathbf{p}| = |\lambda|\mathbf{p} = \mathbf{p} \implies \lambda = 1,$$

and thus 1 is the only eigenvalue of $\mathbf{A}$ on the spectral circle. Now suppose that $index(1) = m > 1$. It follows that $\left\|\mathbf{A}^k\right\|_\infty \to \infty$ as $k \to \infty$ because there is an $m \times m$ Jordan block $\mathbf{J}_\star$ in the Jordan form $\mathbf{J} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P}$ that looks like (7.10.30) on p. 629, so $\left\|\mathbf{J}_\star^k\right\|_\infty \to \infty$, which in turn means that $\left\|\mathbf{J}^k\right\|_\infty \to \infty$ and, consequently, $\left\|\mathbf{J}^k\right\|_\infty = \left\|\mathbf{P}^{-1}\mathbf{A}^k\mathbf{P}\right\|_\infty \le \left\|\mathbf{P}^{-1}\right\|_\infty \left\|\mathbf{A}^k\right\|_\infty \left\|\mathbf{P}\right\|_\infty$ implies

$$\left\|\mathbf{A}^k\right\|_\infty \ge \frac{\left\|\mathbf{J}^k\right\|_\infty}{\left\|\mathbf{P}^{-1}\right\|_\infty \left\|\mathbf{P}\right\|_\infty} \to \infty.$$

Let $\mathbf{A}^k = \left[a_{ij}^{(k)}\right]$, and let i_k denote the row index for which $\left\|\mathbf{A}^k\right\|_\infty = \sum_j a_{i_k j}^{(k)}$. We know that there exists a vector $\mathbf{p} > \mathbf{0}$ such that $\mathbf{p} = \mathbf{A}\mathbf{p}$, so for such an eigenvector,

$$\|\mathbf{p}\|_\infty \ge p_{i_k} = \sum_j a_{i_k j}^{(k)} p_j \ge \left(\sum_j a_{i_k j}^{(k)}\right)(\min_i p_i) = \left\|\mathbf{A}^k\right\|_\infty (\min_i p_i) \to \infty.$$

But this is impossible because $\mathbf{p}$ is a constant vector, so the supposition that $index(1) > 1$ must be false, and thus $index(1) = 1$. ∎

Establishing that $\rho(\mathbf{A})$ is a semisimple eigenvalue of $\mathbf{A} > \mathbf{0}$ was just a steppingstone (but an important one) to get to the following theorem concerning the multiplicities of $\rho(\mathbf{A})$.

Multiplicities of $\rho(\mathbf{A})$

If $\mathbf{A}_{n\times n} > \mathbf{0}$, then $alg\ mult_{\mathbf{A}}(\rho(\mathbf{A})) = 1$. In other words, the spectral radius of $\mathbf{A}$ is a *simple* eigenvalue of $\mathbf{A}$.

So $\dim N(\mathbf{A} - \rho(\mathbf{A})\mathbf{I}) = geo\ mult_{\mathbf{A}}(\rho(\mathbf{A})) = alg\ mult_{\mathbf{A}}(\rho(\mathbf{A})) = 1$.

Proof. As before, assume without loss of generality that $\rho(\mathbf{A}) = 1$, and suppose that $alg\ mult_{\mathbf{A}}(\lambda = 1) = m > 1$. We already know that $\lambda = 1$ is a semisimple eigenvalue, which means that $alg\ mult_{\mathbf{A}}(1) = geo\ mult_{\mathbf{A}}(1)$ (p. 510), so there are m linearly independent eigenvectors associated with $\lambda = 1$. If $\mathbf{x}$ and $\mathbf{y}$ are a pair of independent eigenvectors associated with $\lambda = 1$, then $\mathbf{x} \ne \alpha\mathbf{y}$ for all $\alpha \in \mathcal{C}$. Select a nonzero component from $\mathbf{y}$, say $y_i \ne 0$, and set $\mathbf{z} = \mathbf{x} - (x_i/y_i)\mathbf{y}$. Since $\mathbf{A}\mathbf{z} = \mathbf{z}$, we know from (8.2.7) on p. 663 that $\mathbf{A}|\mathbf{z}| = |\mathbf{z}| > \mathbf{0}$. But this contradicts the fact that $z_i = x_i - (x_i/y_i)y_i = 0$. Therefore, the supposition that $m > 1$ must be false, and thus $m = 1$. ∎

Since $N(\mathbf{A} - \rho(\mathbf{A})\mathbf{I})$ is a one-dimensional space that can be spanned by some $\mathbf{v} > 0$, there is a *unique* eigenvector $\mathbf{p} \in N(\mathbf{A} - \rho(\mathbf{A})\mathbf{I})$ such that $\mathbf{p} > \mathbf{0}$ and $\sum_j p_j = 1$ (it's obtained by the normalization $\mathbf{p} = \mathbf{v}/\|\mathbf{v}\|_1$—see Exercise 8.2.3). This special eigenvector $\mathbf{p}$ is called the ***Perron vector*** for $\mathbf{A} > \mathbf{0}$, and the associated eigenvalue $r = \rho(\mathbf{A})$ is called the ***Perron root*** of $\mathbf{A}$.

Since $\mathbf{A} > \mathbf{0} \iff \mathbf{A}^T > \mathbf{0}$, and since $\rho(\mathbf{A}) = \rho(\mathbf{A}^T)$, it's clear that if $\mathbf{A} > \mathbf{0}$, then in addition to the Perron eigenpair $(r, \mathbf{p})$ for $\mathbf{A}$ there is a corresponding Perron eigenpair $(r, \mathbf{q})$ for $\mathbf{A}^T$. Because $\mathbf{q}^T\mathbf{A} = r\mathbf{q}^T$, the vector $\mathbf{q}^T > \mathbf{0}$ is called the ***left-hand Perron vector*** for $\mathbf{A}$.

While eigenvalues of $\mathbf{A} > \mathbf{0}$ other than $\rho(\mathbf{A})$ may or may not be positive, it turns out that no eigenvectors other than positive multiples of the Perron vector can be positive—or even nonnegative.

No Other Positive Eigenvectors

There are no nonnegative eigenvectors for $\mathbf{A}_{n\times n} > \mathbf{0}$ other than the Perron vector $\mathbf{p}$ and its positive multiples. (8.2.10)

Proof. If $(\lambda, \mathbf{y})$ is an eigenpair for $\mathbf{A}$ such that $\mathbf{y} \geq \mathbf{0}$, and if $\mathbf{x} > \mathbf{0}$ is the Perron vector for $\mathbf{A}^T$, then $\mathbf{x}^T\mathbf{y} > 0$ by (8.2.2), so

$$\rho(\mathbf{A})\mathbf{x}^T = \mathbf{x}^T\mathbf{A} \implies \rho(\mathbf{A})\mathbf{x}^T\mathbf{y} = \mathbf{x}^T\mathbf{A}\mathbf{y} = \lambda\mathbf{x}^T\mathbf{y} \implies \rho(\mathbf{A}) = \lambda. \quad \blacksquare$$

In 1942 the German mathematician Lothar Collatz (1910–1990) discovered the following formula for the Perron root, and in 1950 Helmut Wielandt (p. 534) used it to develop the Perron–Frobenius theory.

Collatz–Wielandt Formula

The Perron root of $\mathbf{A}_{n\times n} > \mathbf{0}$ is given by $r = \max_{\mathbf{x}\in\mathcal{N}} f(\mathbf{x})$, where

$$f(\mathbf{x}) = \min_{\substack{1\leq i\leq n\\ x_i\neq 0}} \frac{[\mathbf{A}\mathbf{x}]_i}{x_i} \quad \text{and} \quad \mathcal{N} = \{\mathbf{x} \,|\, \mathbf{x} \geq \mathbf{0} \text{ with } \mathbf{x} \neq \mathbf{0}\}.$$

Proof. If $\xi = f(\mathbf{x})$ for $\mathbf{x} \in \mathcal{N}$, then $\mathbf{0} \leq \xi\mathbf{x} \leq \mathbf{A}\mathbf{x}$. Let $\mathbf{p}$ and $\mathbf{q}^T$ be the respective the right-hand and left-hand Perron vectors for $\mathbf{A}$ associated with the Perron root r, and use (8.2.3) along with $\mathbf{q}^T\mathbf{x} > 0$ (by (8.2.2)) to write

$$\xi\mathbf{x} \leq \mathbf{A}\mathbf{x} \implies \xi\mathbf{q}^T\mathbf{x} \leq \mathbf{q}^T\mathbf{A}\mathbf{x} = r\mathbf{q}^T\mathbf{x} \implies \xi \leq r \implies f(\mathbf{x}) \leq r \ \ \forall\, \mathbf{x} \in \mathcal{N}.$$

Since $f(\mathbf{p}) = r$ and $\mathbf{p} \in \mathcal{N}$, it follows that $r = \max_{\mathbf{x}\in\mathcal{N}} f(\mathbf{x})$. $\blacksquare$

Below is a summary of the results obtained in this section.

Perron's Theorem

If $\mathbf{A}_{n\times n} > \mathbf{0}$ with $r = \rho(\mathbf{A})$, then the following statements are true.

- $r > 0.$ (8.2.11)
- $r \in \sigma(\mathbf{A})$ (r is called the ***Perron root***). (8.2.12)
- $alg\ mult_{\mathbf{A}}(r) = 1.$ (8.2.13)
- There exists an eigenvector $\mathbf{x} > \mathbf{0}$ such that $\mathbf{A}\mathbf{x} = r\mathbf{x}$. (8.2.14)
- The ***Perron vector*** is the unique vector defined by

$$\mathbf{A}\mathbf{p} = r\mathbf{p}, \qquad \mathbf{p} > \mathbf{0}, \qquad \text{and} \qquad \|\mathbf{p}\|_1 = 1,$$

  and, except for positive multiples of $\mathbf{p}$, there are no other nonnegative eigenvectors for $\mathbf{A}$, regardless of the eigenvalue.
- r is the only eigenvalue on the spectral circle of $\mathbf{A}$. (8.2.15)
- $r = \max_{\mathbf{x}\in\mathcal{N}} f(\mathbf{x})$ (***the Collatz–Wielandt formula***),

$$\text{where } f(\mathbf{x}) = \min_{\substack{1\le i\le n \\ x_i\neq 0}} \frac{[\mathbf{A}\mathbf{x}]_i}{x_i} \text{ and } \mathcal{N} = \{\mathbf{x} \mid \mathbf{x} \geq \mathbf{0} \text{ with } \mathbf{x} \neq \mathbf{0}\}.$$

Note: Our development is the reverse of that of Wielandt and others in the sense that we first proved the existence of the Perron eigenpair $(r, \mathbf{p})$ without reference to $f(\mathbf{x})$, and then we used the Perron eigenpair to established the Collatz-Wielandt formula. Wielandt's approach is to do things the other way around—first prove that $f(\mathbf{x})$ attains a maximum value on $\mathcal{N}$, and then establish existence of the Perron eigenpair by proving that $\max_{\mathbf{x}\in\mathcal{N}} f(\mathbf{x}) = \rho(\mathbf{A})$ with the maximum value being attained at a positive eigenvector $\mathbf{p}$.

Exercises for section 8.2

8.2.1. Verify Perron's theorem by by computing the eigenvalues and eigenvectors for

$$\mathbf{A} = \begin{pmatrix} 7 & 2 & 3 \\ 1 & 8 & 3 \\ 1 & 2 & 9 \end{pmatrix}.$$

Find the right-hand Perron vector $\mathbf{p}$ as well as the left-hand Perron vector $\mathbf{q}^T$.

8.2.2. Convince yourself that (8.2.2)–(8.2.5) are indeed true.

8.2.3. Provide the details that explain why the Perron vector is uniquely defined.

8.2.4. Find the Perron root and the Perron vector for

$$\mathbf{A} = \begin{pmatrix} 1-\alpha & \beta \\ \alpha & 1-\beta \end{pmatrix},$$

where $\alpha + \beta = 1$ with $\alpha, \beta > 0$.

8.2.5. Suppose that $\mathbf{A}_{n \times n} > \mathbf{0}$ has $\rho(\mathbf{A}) = r$.
(a) Explain why $\lim_{k\to\infty} (\mathbf{A}/r)^k$ exists.
(b) Explain why $\lim_{k\to\infty} (\mathbf{A}/r)^k = \mathbf{G} > \mathbf{0}$ is the projector onto $N(\mathbf{A} - r\mathbf{I})$ along $R(\mathbf{A} - r\mathbf{I})$.
(c) Explain why $rank\,(\mathbf{G}) = 1$.

8.2.6. Prove that if every row (or column) sum of $\mathbf{A}_{n \times n} > \mathbf{0}$ is equal to ρ, then $\rho\,(\mathbf{A}) = \rho$.

8.2.7. Prove that if $\mathbf{A}_{n \times n} > \mathbf{0}$, then

$$\min_i \sum_{j=1}^{n} a_{ij} \le \rho\,(\mathbf{A}) \le \max_i \sum_{j=1}^{n} a_{ij}.$$

Hint: Recall Example 7.10.2 (p. 619).

8.2.8. To show the extent to which the hypothesis of positivity cannot be relaxed in Perron's theorem, construct examples of square matrices $\mathbf{A}$ such that $\mathbf{A} \ge \mathbf{0}$, but $\mathbf{A} \not> \mathbf{0}$ (i.e., $\mathbf{A}$ has at least one zero entry), with $r = \rho\,(\mathbf{A}) \in \sigma\,(\mathbf{A})$ that demonstrate the validity of the following statements. Different examples may be used for the different statements.
(a) r can be 0.
(b) $alg\ mult_{\mathbf{A}}\,(r)$ can be greater than 1.
(c) $index\,(r)$ can be greater than 1.
(d) $N(\mathbf{A} - r\mathbf{I})$ need not contain a positive eigenvector.
(e) r need not be the only eigenvalue on the spectral circle.

8.2.9. Establish the min-max version of the Collatz–Wielandt formula that says the Perron root for $\mathbf{A} > \mathbf{0}$ is given by $r = \min_{\mathbf{x}\in\mathcal{P}} g(\mathbf{x})$, where

$$g(\mathbf{x}) = \max_{1\le i\le n} \frac{[\mathbf{A}\mathbf{x}]_i}{x_i} \quad \text{and} \quad \mathcal{P} = \{\mathbf{x} \,|\, \mathbf{x} > \mathbf{0}\}.$$

8.2.10. Notice that $\mathcal{N} = \{\mathbf{x} \,|\, \mathbf{x} \ge \mathbf{0} \text{ with } \mathbf{x} \ne \mathbf{0}\}$ is used in the max-min version of the Collatz–Wielandt formula on p. 666, but $\mathcal{P} = \{\mathbf{x} \,|\, \mathbf{x} > \mathbf{0}\}$ is used in the min-max version in Exercise 8.2.9. Give an example of a matrix $\mathbf{A} > \mathbf{0}$ that shows $r \ne \min_{\mathbf{x}\in\mathcal{N}} g(\mathbf{x})$ when $g(\mathbf{x})$ is defined as

$$g(\mathbf{x}) = \max_{\substack{1\le i\le n \\ x_i\ne 0}} \frac{[\mathbf{A}\mathbf{x}]_i}{x_i}.$$

8.3 NONNEGATIVE MATRICES

Now let zeros creep into the picture and investigate the extent to which Perron's results generalize to nonnegative matrices containing at least one zero entry. The first result along these lines shows how to extend the statements on p. 663 to nonnegative matrices by sacrificing the existence of a positive eigenvector for a nonnegative one.

Nonnegative Eigenpair

For $\mathbf{A}_{n\times n} \geq \mathbf{0}$ with $r = \rho(\mathbf{A})$, the following statements are true.

- $r \in \sigma(\mathbf{A})$, (but $r = 0$ is possible). (8.3.1)
- $\mathbf{Az} = r\mathbf{z}$ for some $\mathbf{z} \in \mathcal{N} = \{\mathbf{x} \,|\, \mathbf{x} \geq \mathbf{0} \text{ with } \mathbf{x} \neq \mathbf{0}\}$. (8.3.2)
- $r = \max_{\mathbf{x}\in\mathcal{N}} f(\mathbf{x})$, where $f(\mathbf{x}) = \min\limits_{\substack{1\leq i\leq n\\ x_i\neq 0}} \dfrac{[\mathbf{Ax}]_i}{x_i}$ (8.3.3)

  (i.e., the Collatz–Wielandt formula remains valid).

Proof. Consider the sequence of positive matrices $\mathbf{A}_k = \mathbf{A} + (1/k)\mathbf{E} > \mathbf{0}$, where $\mathbf{E}$ is the matrix of all 1's, and let $r_k > 0$ and $\mathbf{p}_k > \mathbf{0}$ denote the Perron root and Perron vector for $\mathbf{A}_k$, respectively. Observe that $\{\mathbf{p}_k\}_{k=1}^{\infty}$ is a bounded set because it's contained in the unit 1-sphere in $\Re^n$. The Bolzano–Weierstrass theorem states that each bounded sequence in $\Re^n$ has a convergent subsequence. Therefore, $\{\mathbf{p}_k\}_{k=1}^{\infty}$ has convergent subsequence

$$\{\mathbf{p}_{k_i}\}_{i=1}^{\infty} \to \mathbf{z}, \text{ where } \mathbf{z} \geq \mathbf{0} \text{ with } \mathbf{z} \neq \mathbf{0} \text{ (because } \mathbf{p}_{k_i} > \mathbf{0} \text{ and } \|\mathbf{p}_{k_i}\|_1 = 1).$$

Since $\mathbf{A}_1 > \mathbf{A}_2 > \cdots > \mathbf{A}$, the result in Example 7.10.2 (p. 619) guarantees that $r_1 \geq r_2 \geq \cdots \geq r$, so $\{r_k\}_{k=1}^{\infty}$ is a monotonic sequence of positive numbers that is bounded below by r. A standard result from analysis guarantees that

$$\lim_{k\to\infty} r_k = r^\star \text{ exists, and } r^\star \geq r. \quad \text{In particular,} \quad \lim_{i\to\infty} r_{k_i} = r^\star \geq r.$$

But $\lim_{k\to\infty} \mathbf{A}_k = \mathbf{A}$ implies $\lim_{i\to\infty} \mathbf{A}_{k_i} \to \mathbf{A}$, so, by using the easily established fact that the limit of a product is the product of the limits (provided that all limits exist), it's also true that

$$\mathbf{Az} = \lim_{i\to\infty} \mathbf{A}_{k_i}\mathbf{p}_{k_i} = \lim_{i\to\infty} r_{k_i}\mathbf{p}_{k_i} = r^\star\mathbf{z} \implies r^\star \in \sigma(\mathbf{A}) \implies r^\star \leq r.$$

Consequently, $r^\star = r$, and $\mathbf{Az} = r\mathbf{z}$ with $\mathbf{z} \geq \mathbf{0}$ and $\mathbf{z} \neq \mathbf{0}$. Thus (8.3.1) and (8.3.2) are proven. To prove (8.3.3), let $\mathbf{q}_k^T > \mathbf{0}$ be the left-hand Perron vector of $\mathbf{A}_k$. For every $\mathbf{x} \in \mathcal{N}$ and $k > 0$ we have $\mathbf{q}_k^T\mathbf{x} > 0$ (by (8.2.2)), and

$$\begin{aligned}\mathbf{0} \leq f(\mathbf{x})\mathbf{x} \leq \mathbf{Ax} \leq \mathbf{A}_k\mathbf{x} &\implies f(\mathbf{x})\mathbf{q}_k^T\mathbf{x} \leq \mathbf{q}_k^T\mathbf{A}_k\mathbf{x} = r_k\mathbf{q}_k^T\mathbf{x} \implies f(\mathbf{x}) \leq r_k \\ &\implies f(\mathbf{x}) \leq r \quad (\text{because } r_k \to r^* = r).\end{aligned}$$

Since $f(\mathbf{z}) = r$ and $\mathbf{z} \in \mathcal{N}$, it follows that $\max_{\mathbf{x}\in\mathcal{N}} f(\mathbf{x}) = r$. ∎

This is as far as Perron's theorem can be generalized to nonnegative matrices without additional hypothesis. For example, $\mathbf{A} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ shows that properties (8.2.11), (8.2.13), and (8.2.14) on p. 667 do not hold for general nonnegative matrices, and $\mathbf{A} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ shows that (8.2.15) is also lost. Rather than accepting that the major issues concerning spectral properties of nonnegative matrices had been settled, Frobenius had the insight to look below the surface and see that the problem doesn't stem just from the existence of zero entries, but rather from the *positions* of the zero entries. For example, (8.2.13) and (8.2.14) are false for

$$\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \text{ but they are true for } \widetilde{\mathbf{A}} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}. \tag{8.3.4}$$

Frobenius's genius was to see the difference between $\mathbf{A}$ and $\widetilde{\mathbf{A}}$ in terms of reducibility and to relate these ideas to spectral properties of nonnegative matrices. Reducibility and graphs were discussed in Example 4.4.6 (p. 202) and Exercise 4.4.20 (p. 209), but for the sake of continuity they are reviewed below.

Reducibility and Graphs

- $\mathbf{A}_{n\times n}$ is said to be a ***reducible matrix*** when there exists a permutation matrix $\mathbf{P}$ such that

$$\mathbf{P}^T\mathbf{A}\mathbf{P} = \begin{pmatrix} \mathbf{X} & \mathbf{Y} \\ \mathbf{0} & \mathbf{Z} \end{pmatrix}, \quad \text{where } \mathbf{X} \text{ and } \mathbf{Z} \text{ are both square.}$$

  Otherwise $\mathbf{A}$ is said to be an ***irreducible matrix.***

- $\mathbf{P}^T\mathbf{A}\mathbf{P}$ is called a ***symmetric permutation*** of $\mathbf{A}$. The effect is to interchange rows in the same way as columns are interchanged.

- The ***graph*** $\mathcal{G}(\mathbf{A})$ of $\mathbf{A}$ is defined to be the directed graph on n nodes $\{N_1, N_2, \ldots, N_n\}$ in which there is a directed edge leading *from* N_i *to* N_j if and only if $a_{ij} \neq 0$.

- $\mathcal{G}(\mathbf{P}^T\mathbf{A}\mathbf{P}) = \mathcal{G}(\mathbf{A})$ whenever $\mathbf{P}$ is a permutation matrix—the effect is simply a relabeling of nodes.

- $\mathcal{G}(\mathbf{A})$ is called ***strongly connected*** if for each pair of nodes (N_i, N_k) there is a sequence of directed edges leading from N_i to N_k.

- $\mathbf{A}$ is an irreducible matrix if and only if $\mathcal{G}(\mathbf{A})$ is strongly connected (see Exercise 4.4.20 on p. 209).

For example, the matrix $\mathbf{A}$ in (8.3.4) is reducible because

$$\mathbf{P}^T\mathbf{A}\mathbf{P} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{for} \quad \mathbf{P} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

and, as can be seen from Figure 8.3.1, $\mathcal{G}(\mathbf{A})$ is not strongly connected because there is no sequence of paths leading from node 1 to node 2. On the other hand, $\widetilde{\mathbf{A}}$ is irreducible, and as shown in Figure 8.3.1, $\mathcal{G}(\widetilde{\mathbf{A}})$ is strongly connected because each node is accessible from the other.

FIGURE 8.3.1

This discussion suggests that some of Perron's properties given on p. 667 extend to nonnegative matrices when the zeros are in just the right positions to insure irreducibility. To prove that this is in fact the case, the following lemma is needed. It shows how to convert a nonnegative irreducible matrix into a positive matrix in a useful fashion.

Converting Nonnegativity & Irreducibility to Positivity

If $\mathbf{A}_{n\times n} \geq \mathbf{0}$ is irreducible, then $(\mathbf{I}+\mathbf{A})^{n-1} > \mathbf{0}$. (8.3.5)

Proof. Let $a_{ij}^{(k)}$ denote the (i,j)-entry in $\mathbf{A}^k$, and observe that

$$a_{ij}^{(k)} = \sum_{h_1,\ldots,h_{k-1}} a_{ih_1}a_{h_1h_2}\cdots a_{h_{k-1}j} > 0$$

if and only if there exists a set of indicies $h_1, h_2, \ldots, h_{k-1}$ such that

$$a_{ih_1} > 0 \quad \text{and} \quad a_{h_1h_2} > 0 \quad \text{and} \quad \cdots \quad \text{and} \quad a_{h_{k-1}j} > 0.$$

In other words, there is a sequence of k paths $N_i \to N_{h_1} \to N_{h_2} \to \cdots \to N_j$ in $\mathcal{G}(\mathbf{A})$ that lead from node N_i to node N_j if and only if $a_{ij}^{(k)} > 0$. The irreducibility of $\mathbf{A}$ insures that $\mathcal{G}(\mathbf{A})$ is strongly connected, so for any pair of nodes (N_i, N_j) there is a sequence of k paths (with $k < n$) from N_i to N_j. This means that for each position (i,j), there is some $0 \leq k \leq n-1$ such that $a_{ij}^{(k)} > 0$, and this guarantees that for each i and j,

$$\left[(\mathbf{I}+\mathbf{A})^{n-1}\right]_{ij} = \left[\sum_{k=0}^{n-1}\binom{n-1}{k}\mathbf{A}^k\right]_{ij} = \sum_{k=0}^{n-1}\binom{n-1}{k}a_{ij}^{(k)} > \mathbf{0}. \quad \blacksquare$$

With the exception of the Collatz–Wielandt formula, we have seen that $\rho(\mathbf{A}) \in \sigma(\mathbf{A})$ is the only property in the list of Perron properties on p. 667 that extends to nonnegative matrices without additional hypothesis. The next theorem shows how adding irreducibility to nonnegativity recovers the Perron properties (8.2.11), (8.2.13), and (8.2.14).

Perron–Frobenius Theorem

If $\mathbf{A}_{n\times n} \geq \mathbf{0}$ is irreducible, then each of the following is true.

- $r = \rho(\mathbf{A}) \in \sigma(\mathbf{A})$ and $r > 0$. (8.3.6)
- *alg mult*$_{\mathbf{A}}(r) = 1$. (8.3.7)
- There exists an eigenvector $\mathbf{x} > \mathbf{0}$ such that $\mathbf{Ax} = r\mathbf{x}$. (8.3.8)
- The unique vector defined by

$$\mathbf{Ap} = r\mathbf{p}, \qquad \mathbf{p} > \mathbf{0}, \qquad \text{and} \qquad \|\mathbf{p}\|_1 = 1, \tag{8.3.9}$$

  is called the ***Perron vector.*** There are no nonnegative eigenvectors for $\mathbf{A}$ except for positive multiples of $\mathbf{p}$, regardless of the eigenvalue.
- The Collatz–Wielandt formula $r = \max_{\mathbf{x}\in\mathcal{N}} f(\mathbf{x})$,

$$\text{where } f(\mathbf{x}) = \min_{\substack{1\leq i\leq n \\ x_i\neq 0}} \frac{[\mathbf{Ax}]_i}{x_i} \text{ and } \mathcal{N} = \{\mathbf{x} \mid \mathbf{x} \geq \mathbf{0} \text{ with } \mathbf{x} \neq \mathbf{0}\}$$

  was established in (8.3.3) for all nonnegative matrices, but it is included here for the sake of completeness.

Proof. We already know from (8.3.2) that $r = \rho(\mathbf{A}) \in \sigma(\mathbf{A})$. To prove that *alg mult*$_{\mathbf{A}}(r) = 1$, let $\mathbf{B} = (\mathbf{I} + \mathbf{A})^{n-1} > \mathbf{0}$ be the matrix in (8.3.5). It follows from (7.9.3) that $\lambda \in \sigma(\mathbf{A})$ if and only if $(1+\lambda)^{n-1} \in \sigma(\mathbf{B})$, and *alg mult*$_{\mathbf{A}}(\lambda) =$ *alg mult*$_{\mathbf{B}}\left((1+\lambda)^{n-1}\right)$. Consequently, if $\mu = \rho(\mathbf{B})$, then

$$\mu = \max_{\lambda\in\sigma(\mathbf{A})} |(1+\lambda)|^{n-1} = \left\{\max_{\lambda\in\sigma(\mathbf{A})} |(1+\lambda)|\right\}^{n-1} = (1+r)^{n-1}$$

because when a circular disk $|z| \leq \rho$ is translated one unit to the right, the point of maximum modulus in the resulting disk $|z+1| \leq \rho$ is $z = 1+\rho$ (it's clear if you draw a picture). Therefore, *alg mult*$_{\mathbf{A}}(r) = 1$; otherwise *alg mult*$_{\mathbf{B}}(\mu) > 1$, which is impossible because $\mathbf{B} > \mathbf{0}$. To see that $\mathbf{A}$ has a positive eigenvector associated with r, recall from (8.3.2) that there exists a nonnegative eigenvector $\mathbf{x} \geq \mathbf{0}$ associated with r. It's a simple consequence of (7.9.9) that if $(\lambda, \mathbf{x})$ is an eigenpair for $\mathbf{A}$, then $(f(\lambda), \mathbf{x})$ is an eigenpair for $f(\mathbf{A})$ (Exercise 7.9.9, p. 613), so $(r, \mathbf{x})$ being an eigenpair for $\mathbf{A}$ implies that $(\mu, \mathbf{x})$ is an eigenpair for $\mathbf{B}$. Hence (8.2.10) insures that $\mathbf{x}$ must be a positive multiple of the Perron vector of $\mathbf{B}$, and thus $\mathbf{x}$ must in fact be positive. Now, $r > 0$; otherwise $\mathbf{Ax} = \mathbf{0}$, which is impossible because $\mathbf{A} \geq \mathbf{0}$ and $\mathbf{x} > \mathbf{0}$ forces $\mathbf{Ax} > \mathbf{0}$. The argument used to prove (8.2.10) also proves (8.3.9). ■

Example 8.3.1

Problem: Suppose that $\mathbf{A}_{n\times n} \geq \mathbf{0}$ is irreducible with $r = \rho(\mathbf{A})$, and suppose that $r\mathbf{z} \leq \mathbf{Az}$ for $\mathbf{z} \geq \mathbf{0}$. Explain why $r\mathbf{z} = \mathbf{Az}$, and $\mathbf{z} > \mathbf{0}$.

Solution: If $r\mathbf{z} < \mathbf{Az}$, then by using the Perron vector $\mathbf{q} > \mathbf{0}$ for $\mathbf{A}^T$ we have

$$(\mathbf{A} - r\mathbf{I})\mathbf{z} \geq \mathbf{0} \quad \Longrightarrow \quad \mathbf{q}^T(\mathbf{A} - r\mathbf{I})\mathbf{z} > 0,$$

which is impossible since $\mathbf{q}^T(\mathbf{A} - r\mathbf{I}) = \mathbf{0}$. Thus $r\mathbf{z} = \mathbf{Az}$, and since $\mathbf{z}$ must be a multiple of the Perron vector for $\mathbf{A}$ by (8.3.9), we also have that $\mathbf{z} > \mathbf{0}$.

The only property in the list on p. 667 that irreducibility is not able to salvage is (8.2.15), which states that there is only one eigenvalue on the spectral circle. Indeed, $\mathbf{A} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is nonnegative and irreducible, but the eigenvalues ± 1 are both on the unit circle. The property of having (or not having) only one eigenvalue on the spectral circle divides the set of nonnegative irreducible matrices into two important classes.

Primitive Matrices

- A nonnegative irreducible matrix $\mathbf{A}$ having only one eigenvalue, $r = \rho(\mathbf{A})$, on its spectral circle is said to be a ***primitive matrix.***
- A nonnegative irreducible matrix having $h > 1$ eigenvalues on its spectral circle is called ***imprimitive,*** and h is referred to as ***index of imprimitivity.***
- A nonnegative irreducible matrix $\mathbf{A}$ with $r = \rho(\mathbf{A})$ is primitive if and only if $\lim_{k\to\infty}(\mathbf{A}/r)^k$ exists, in which case

$$\lim_{k\to\infty} \left(\frac{\mathbf{A}}{r}\right)^k = \mathbf{G} = \frac{\mathbf{p}\mathbf{q}^T}{\mathbf{q}^T\mathbf{p}} > \mathbf{0}, \tag{8.3.10}$$

where $\mathbf{p}$ and $\mathbf{q}$ are the respective Perron vectors for $\mathbf{A}$ and $\mathbf{A}^T$. $\mathbf{G}$ is the (spectral) projector onto $N(\mathbf{A} - r\mathbf{I})$ along $R(\mathbf{A} - r\mathbf{I})$.

Proof of (8.3.10). The Perron–Frobenius theorem insures that $1 = \rho(\mathbf{A}/r)$ is a simple eigenvalue for $\mathbf{A}/r$, and it's clear that $\mathbf{A}$ is primitive if and only if $\mathbf{A}/r$ is primitive. In other words, $\mathbf{A}$ is primitive if and only if $1 = \rho(\mathbf{A}/r)$ is the only eigenvalue on the unit circle, which is equivalent to saying that $\lim_{k\to\infty}(\mathbf{A}/r)^k$ exists by the results on p. 630. The structure of the limit as described in (8.3.10) is the result of (7.2.12) on p. 518. ∎

The next two results, discovered by Helmut Wielandt (p. 534) in 1950, establish the remarkable fact that the eigenvalues on the spectral circle of an imprimitive matrix are in fact the h^{th} roots of the spectral radius.

Wielandt's Theorem

If $|\mathbf{B}| \leq \mathbf{A}_{n\times n}$, where $\mathbf{A}$ is irreducible, then $\rho(\mathbf{B}) \leq \rho(\mathbf{A})$. If equality holds (i.e., if $\mu = \rho(\mathbf{A})\,\mathrm{e}^{\mathrm{i}\phi} \in \sigma(\mathbf{B})$ for some ϕ), then

$$\mathbf{B} = \mathrm{e}^{\mathrm{i}\phi}\mathbf{D}\mathbf{A}\mathbf{D}^{-1} \quad \text{for some} \quad \mathbf{D} = \begin{pmatrix} \mathrm{e}^{\mathrm{i}\theta_1} & & & \\ & \mathrm{e}^{\mathrm{i}\theta_2} & & \\ & & \ddots & \\ & & & \mathrm{e}^{\mathrm{i}\theta_n} \end{pmatrix}, \tag{8.3.11}$$

and conversely.

Proof. We already know that $\rho(\mathbf{B}) \leq \rho(\mathbf{A})$ by Example 7.10.2 (p. 619). If $\rho(\mathbf{B}) = r = \rho(\mathbf{A})$, and if $(\mu, \mathbf{x})$ is an eigenpair for $\mathbf{B}$ such that $|\mu| = r$, then

$$r|\mathbf{x}| = |\mu|\,|\mathbf{x}| = |\mu\mathbf{x}| = |\mathbf{B}\mathbf{x}| \leq |\mathbf{B}|\,|\mathbf{x}| \leq \mathbf{A}|\mathbf{x}| \quad \Longrightarrow \quad |\mathbf{B}|\,|\mathbf{x}| = r|\mathbf{x}|$$

because the result in Example 8.3.1 insures that $\mathbf{A}|\mathbf{x}| = r|\mathbf{x}|$, and $|\mathbf{x}| > \mathbf{0}$. Consequently, $(\mathbf{A} - |\mathbf{B}|)|\mathbf{x}| = \mathbf{0}$. But $\mathbf{A} - |\mathbf{B}| \geq \mathbf{0}$, and $|\mathbf{x}| > \mathbf{0}$, so $\mathbf{A} = |\mathbf{B}|$ by (8.2.4). Since $x_k/|x_k|$ is on the unit circle, $x_k/|x_k| = \mathrm{e}^{\mathrm{i}\theta_k}$ for some θ_k. Set

$$\mathbf{D} = \begin{pmatrix} \mathrm{e}^{\mathrm{i}\theta_1} & & & \\ & \mathrm{e}^{\mathrm{i}\theta_2} & & \\ & & \ddots & \\ & & & \mathrm{e}^{\mathrm{i}\theta_n} \end{pmatrix}, \quad \text{and notice that} \quad \mathbf{x} = \mathbf{D}|\mathbf{x}|.$$

Since $|\mu| = r$, there is a $\phi \in \Re$ such that $\mu = r\mathrm{e}^{\mathrm{i}\phi}$, and hence

$$\mathbf{B}\mathbf{D}|\mathbf{x}| = \mathbf{B}\mathbf{x} = \mu\mathbf{x} = r\mathrm{e}^{\mathrm{i}\phi}\mathbf{x} = r\mathrm{e}^{\mathrm{i}\phi}\mathbf{D}|\mathbf{x}| \;\Rightarrow\; \mathrm{e}^{-\mathrm{i}\phi}\mathbf{D}^{-1}\mathbf{B}\mathbf{D}|\mathbf{x}| = r|\mathbf{x}| = \mathbf{A}|\mathbf{x}|. \tag{8.3.12}$$

For convenience, let $\mathbf{C} = \mathrm{e}^{-\mathrm{i}\phi}\mathbf{D}^{-1}\mathbf{B}\mathbf{D}$, and note that $|\mathbf{C}| = |\mathbf{B}| = \mathbf{A}$ to write (8.3.12) as $\mathbf{0} = (|\mathbf{C}| - \mathbf{C})|\mathbf{x}|$. Considering only the real part of this equation yields $\mathbf{0} = \big(|\mathbf{C}| - \text{Re}\,(\mathbf{C})\big)|\mathbf{x}|$. But $|\mathbf{C}| \geq \text{Re}\,(\mathbf{C})$, and $|\mathbf{x}| > \mathbf{0}$, so it follows from (8.2.4) that $\text{Re}\,(\mathbf{C}) = |\mathbf{C}|$, and hence

$$\text{Re}\,(c_{ij}) = |c_{ij}| = \sqrt{\text{Re}\,(c_{ij})^{\,2} + \text{Im}\,(c_{ij})^{\,2}} \implies \text{Im}\,(c_{ij}) = 0 \implies \text{Im}\,(\mathbf{C}) = \mathbf{0}.$$

Therefore, $\mathbf{C} = \text{Re}\,(\mathbf{C}) = |\mathbf{C}| = \mathbf{A}$, which implies $\mathbf{B} = \mathrm{e}^{\mathrm{i}\phi}\mathbf{DAD}^{-1}$. Conversely, if $\mathbf{B} = \mathrm{e}^{\mathrm{i}\phi}\mathbf{DAD}^{-1}$, then similarity insures that $\rho(\mathbf{B}) = \rho\left(\mathrm{e}^{\mathrm{i}\phi}\mathbf{A}\right) = \rho(\mathbf{A})$. ■

h^{th} Roots of $\rho(\mathbf{A})$ on Spectral Circle

If $\mathbf{A}_{n\times n} \geq \mathbf{0}$ is irreducible and has h eigenvalues $\{\lambda_1, \lambda_2, \ldots, \lambda_h\}$ on its spectral circle, then each of the following statements is true.

- *alg mult*$_{\mathbf{A}}(\lambda_k) = 1$ for $k = 1, 2, \ldots, h$. (8.3.13)
- $\{\lambda_1, \lambda_2, \ldots, \lambda_h\}$ are the h^{th} roots of $r = \rho(\mathbf{A})$ given by

$$\{r,\, r\omega,\, r\omega^2,\, \ldots,\, r\omega^{h-1}\}, \quad \text{where} \quad \omega = \mathrm{e}^{2\pi\mathrm{i}/h}. \tag{8.3.14}$$

Proof. Let $\mathcal{S} = \{r, r\mathrm{e}^{\mathrm{i}\theta_1}, \ldots, r\mathrm{e}^{\mathrm{i}\theta_{h-1}}\}$ denote the eigenvalues on the spectral circle of $\mathbf{A}$. Applying (8.3.11) with $\mathbf{B} = \mathbf{A}$ and $\mu = r\mathrm{e}^{\mathrm{i}\theta_k}$ insures the existence of a diagonal matrix $\mathbf{D}_k$ such that $\mathbf{A} = \mathrm{e}^{\mathrm{i}\theta_k}\mathbf{D}_k\mathbf{A}\mathbf{D}_k^{-1}$, thus showing that $\mathrm{e}^{\mathrm{i}\theta_k}\mathbf{A}$ is similar to $\mathbf{A}$. Since r is a simple eigenvalue of $\mathbf{A}$ (by the Perron–Frobenius theorem), $r\mathrm{e}^{\mathrm{i}\theta_k}$ must be a simple eigenvalue of $\mathrm{e}^{\mathrm{i}\theta_k}\mathbf{A}$. But similarity transformations preserve eigenvalues and algebraic multiplicities (because the Jordan structure is preserved), so $r\mathrm{e}^{\mathrm{i}\theta_k}$ must be a simple eigenvalue of $\mathbf{A}$, thus establishing (8.3.13). To prove (8.3.14), consider another eigenvalue $r\mathrm{e}^{\mathrm{i}\theta_s} \in \mathcal{S}$. Again, we can write $\mathbf{A} = \mathrm{e}^{\mathrm{i}\theta_s}\mathbf{D}_s\mathbf{A}\mathbf{D}_s^{-1}$ for some $\mathbf{D}_s$, so

$$\mathbf{A} = \mathrm{e}^{\mathrm{i}\theta_k}\mathbf{D}_k\mathbf{A}\mathbf{D}_k^{-1} = \mathrm{e}^{\mathrm{i}\theta_k}\mathbf{D}_k(\mathrm{e}^{\mathrm{i}\theta_s}\mathbf{D}_s\mathbf{A}\mathbf{D}_s^{-1})\mathbf{D}_k^{-1} = \mathrm{e}^{\mathrm{i}(\theta_k+\theta_r)}(\mathbf{D}_k\mathbf{D}_s)\mathbf{A}(\mathbf{D}_k\mathbf{D}_s)^{-1}$$

and, consequently, $r\mathrm{e}^{\mathrm{i}(\theta_k+\theta_r)}$ is also an eigenvalue on the spectral circle of $\mathbf{A}$. In other words, $\mathcal{S} = \{r, r\mathrm{e}^{\mathrm{i}\theta_1}, \ldots, r\mathrm{e}^{\mathrm{i}\theta_{h-1}}\}$ is closed under multiplication. This means that $\mathcal{G} = \{1, \mathrm{e}^{\mathrm{i}\theta_1}, \ldots, \mathrm{e}^{\mathrm{i}\theta_{h-1}}\}$ is closed under multiplication, and it follows that $\mathcal{G}$ is a finite commutative group of order h. A standard result from algebra states that the h^{th} power of every element in a finite group of order h must be the identity element in the group. Therefore, $(\mathrm{e}^{\mathrm{i}\theta_k})^h = 1$ for each $0 \leq k \leq h-1$, so $\mathcal{G}$ is the set of the h^{th} roots of unity $\mathrm{e}^{2\pi k\mathrm{i}/h}$ $(0 \leq k \leq h-1)$, and thus $\mathcal{S}$ must be the h^{th} roots of r. ■

Combining the preceding results reveals just how special the spectrum of an imprimitive matrix is.

Rotational Invariance

If $\mathbf{A}$ is imprimitive with h eigenvalues on its spectral circle, then $\sigma(\mathbf{A})$ is invariant under rotation about the origin through an angle $2\pi/h$. No rotation less than $2\pi/h$ can preserve $\sigma(\mathbf{A})$. (8.3.15)

Proof. Since $\lambda \in \sigma(\mathbf{A}) \iff \lambda e^{2\pi i/h} \in \sigma(e^{2\pi i/h}\mathbf{A})$, it follows that $\sigma(e^{2\pi i/h}\mathbf{A})$ is $\sigma(\mathbf{A})$ rotated through $2\pi/h$. But (8.3.11) and (8.3.14) insure that $\mathbf{A}$ and $e^{2\pi i/h}\mathbf{A}$ are similar and, consequently, $\sigma(\mathbf{A}) = \sigma(e^{2\pi i/h}\mathbf{A})$. No rotation less than $2\pi/h$ can keep $\sigma(\mathbf{A})$ invariant because (8.3.14) makes it clear that the eigenvalues on the spectral circle won't go back into themselves for rotations less than $2\pi/h$. ■

Example 8.3.2

The Spectral Projector Is Positive. We already know from (8.3.10) that if $\mathbf{A}$ is a primitive matrix, and if $\mathbf{G}$ is the spectral projector associated with $r = \rho(\mathbf{A})$, then $\mathbf{G} > \mathbf{0}$.

Problem: Explain why this is also true for an imprimitive matrix. In other words, establish the fact that *if* $\mathbf{G}$ *is the spectral projector associated with* $r = \rho(\mathbf{A})$ *for any nonnegative irreducible matrix* $\mathbf{A}$, *then* $\mathbf{G} > \mathbf{0}$.

Solution: Being imprimitive means that $\mathbf{A}$ is nonnegative and irreducible with more than one eigenvalue on the spectral circle. However, (8.3.13) says that each eigenvalue on the spectral circle is simple, so the results concerning Cesàro summability on p. 633 can be applied to $\mathbf{A}/r$ to conclude that

$$\lim_{k\to\infty} \frac{\mathbf{I} + (\mathbf{A}/r) + \cdots + (\mathbf{A}/r)^{k-1}}{k} = \mathbf{G},$$

where $\mathbf{G}$ is the spectral projector onto $N((\mathbf{A}/r) - \mathbf{I}) = N(\mathbf{A} - r\mathbf{I})$ along $R((\mathbf{A}/r) - \mathbf{I}) = R(\mathbf{A} - r\mathbf{I})$. Since r is a simple eigenvalue the same argument used to establish (8.3.10) (namely, invoking (7.2.12) on p. 518) shows that

$$\mathbf{G} = \frac{\mathbf{p}\mathbf{q}^T}{\mathbf{q}^T\mathbf{p}} > \mathbf{0},$$

where $\mathbf{p}$ and $\mathbf{q}$ are the respective Perron vectors for $\mathbf{A}$ and $\mathbf{A}^T$.

Trying to determine if an irreducible matrix $\mathbf{A} \geq \mathbf{0}$ is primitive or imprimitive by finding the eigenvalues is generally a difficult task, so it's natural to ask if there's another way. It turns out that there is, and, as the following example shows, determining primitivity can sometimes be trivial.

Example 8.3.3

Sufficient Condition for Primitivity. If a nonnegative irreducible matrix $\mathbf{A}$ has at least one positive diagonal element, then $\mathbf{A}$ is primitive.

Proof. Suppose there are $h > 1$ eigenvalues on the spectral circle. We know from (8.3.15) that if $\lambda_0 \in \sigma(\mathbf{A})$, then $\lambda_k = \lambda_0 e^{2\pi i k/h} \in \sigma(\mathbf{A})$ for $k = 0, 1, \ldots, h-1$, so

$$\sum_{k=0}^{h-1} \lambda_k = \lambda_0 \sum_{k=0}^{h-1} e^{2\pi i k/h} = 0 \quad \text{(roots of unity sum to 1—see p. 357).}$$

This implies that the sum of *all* of the eigenvalues is zero. In other words,

- if $\mathbf{A}$ is imprimitive, then $trace(\mathbf{A}) = 0$. (Recall (7.1.7) on p. 494.)

Therefore, if $\mathbf{A}$ has a positive diagonal entry, then $\mathbf{A}$ must be primitive.

Another of Frobenius's contributions was to show how the powers of a nonnegative matrix determine whether or not the matrix is primitive. The exact statement is as follows.

Frobenius's Test for Primitivity

$\mathbf{A} \geq \mathbf{0}$ is primitive if and only if $\mathbf{A}^m > \mathbf{0}$ for some $m > 0$. (8.3.16)

Proof. First assume that $\mathbf{A}^m > \mathbf{0}$ for some m. This implies that $\mathbf{A}$ is irreducible; otherwise there exists a permutation matrix such that

$$\mathbf{A} = \mathbf{P}\begin{pmatrix} \mathbf{X} & \mathbf{Y} \\ \mathbf{0} & \mathbf{Z} \end{pmatrix}\mathbf{P}^T \implies \mathbf{A}^m = \mathbf{P}\begin{pmatrix} \mathbf{X}^m & \star \\ \mathbf{0} & \mathbf{Z}^m \end{pmatrix}\mathbf{P}^T \text{ has zero entries.}$$

Suppose that $\mathbf{A}$ has h eigenvalues $\{\lambda_1, \lambda_2, \ldots, \lambda_h\}$ on its spectral circle so that $r = \rho(\mathbf{A}) = |\lambda_1| = \cdots = |\lambda_h| > |\lambda_{h+1}| > \cdots > |\lambda_n|$. Since $\lambda \in \sigma(\mathbf{A})$ implies $\lambda^m \in \sigma(\mathbf{A}^m)$ with $alg\ mult_{\mathbf{A}}(\lambda) = alg\ mult_{\mathbf{A}^m}(\lambda^m)$ (consider the Jordan form—Exercise 7.9.9 on p. 613), it follows that λ_k^m $(1 \leq k \leq h)$ is on the spectral circle of $\mathbf{A}^m$ with $alg\ mult_{\mathbf{A}}(\lambda_k) = alg\ mult_{\mathbf{A}^m}(\lambda_k^m)$. Perron's theorem (p. 667) insures that $\mathbf{A}^m$ has only one eigenvalue (which must be r^m) on its spectral circle, so $r^m = \lambda_1^m = \lambda_2^m = \cdots = \lambda_h^m$. But this means that

$$alg\ mult_{\mathbf{A}}(r) = alg\ mult_{\mathbf{A}^m}(r^m) = h,$$

and therefore $h = 1$ by (8.3.7). Conversely, if $\mathbf{A}$ is primitive with $r = \rho(\mathbf{A})$, then $\lim_{k\to\infty}(\mathbf{A}/r)^k > \mathbf{0}$ by (8.3.10). Hence there must be some m such that $(\mathbf{A}/r)^m > \mathbf{0}$, and thus $\mathbf{A}^m > \mathbf{0}$. ■

Example 8.3.4

Suppose that we wish to decide whether or not a nonnegative matrix $\mathbf{A}$ is primitive by computing the sequence of powers $\mathbf{A}, \mathbf{A}^2, \mathbf{A}^3, \ldots$. Since this can be a laborious task, it would be nice to know when we have computed enough powers of $\mathbf{A}$ to render a judgement. Unfortunately there is nothing in the statement or proof of Frobenius's test to help us with this decision. But Wielandt provided an answer by proving that a nonnegative matrix $\mathbf{A}_{n\times n}$ is primitive if and only if $\mathbf{A}^{n^2-2n+2} > \mathbf{0}$. Furthermore, $n^2 - 2n + 2$ is the smallest such exponent that works for the class of $n \times n$ primitive matrices having all zeros on the diagonal—see Exercise 8.3.9.

Problem: Determine whether or not $\mathbf{A} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 3 & 4 & 0 \end{pmatrix}$ is primitive.

Solution: Since $\mathbf{A}$ has zeros on the diagonal, the result in Example 8.3.3 doesn't apply, so we are forced into computing powers of $\mathbf{A}$. This job is simplified by noticing that if $\mathbf{B} = \beta(\mathbf{A})$ is the Boolean matrix that results from setting

$$b_{ij} = \begin{cases} 1 & \text{if } a_{ij} > 0, \\ 0 & \text{if } a_{ij} = 0, \end{cases}$$

then $[\mathbf{B}^k]_{ij} > 0$ if and only if $[\mathbf{A}^k]_{ij} > 0$ for every $k > 0$. This means that instead of using $\mathbf{A}, \mathbf{A}^2, \mathbf{A}^3, \ldots$ to decide on primitivity, we need only compute

$$\mathbf{B}_1 = \beta(\mathbf{A}), \quad \mathbf{B}_2 = \beta(\mathbf{B}_1\mathbf{B}_1), \quad \mathbf{B}_3 = \beta(\mathbf{B}_1\mathbf{B}_2), \quad \mathbf{B}_4 = \beta(\mathbf{B}_1\mathbf{B}_3), \ldots,$$

going no further than $\mathbf{B}_{n^2-2n+2}$, and these computations require only Boolean operations **AND** and **OR**. The matrix $\mathbf{A}$ in this example is primitive because

$$\mathbf{B}_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \mathbf{B}_2 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \mathbf{B}_3 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \mathbf{B}_4 = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \mathbf{B}_5 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

The powers of an irreducible matrix $\mathbf{A} \geq \mathbf{0}$ can tell us if $\mathbf{A}$ has more than one eigenvalue on its spectral circle, but the powers of $\mathbf{A}$ provide no clue to the number of such eigenvalues. The next theorem shows how the index of imprimitivity can be determined without explicitly calculating the eigenvalues.

Index of Imprimitivity

If $c(x) = x^n + c_{k_1}x^{n-k_1} + c_{k_2}x^{n-k_2} + \cdots + c_{k_s}x^{n-k_s} = 0$ is the characteristic equation of an imprimitive matrix $\mathbf{A}_{n\times n}$ in which only the terms with nonzero coefficients are listed (i.e., each $c_{k_j} \neq 0$, and $n > (n-k_1) > \cdots > (n-k_s)$), then the index of imprimitivity h is the greatest common divisor of $\{k_1, k_2, \ldots, k_s\}$.

Proof. We know from (8.3.15) that if $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ are the eigenvalues of $\mathbf{A}$ (including multiplicities), then $\{\omega\lambda_1, \omega\lambda_2, \ldots, \omega\lambda_n\}$ are also the eigenvalues of $\mathbf{A}$, where $\omega = e^{2\pi i/h}$. It follows from the results on p. 494 that

$$c_{k_j} = (-1)^{k_j} \sum_{1 \le i_1 < \cdots < i_{k_j} \le n} \lambda_{i_1} \cdots \lambda_{i_{k_j}} = (-1)^{k_j} \sum_{1 \le i_1 < \cdots < i_{k_j} \le n} \omega\lambda_{i_1} \cdots \omega\lambda_{i_{k_j}} = \omega^{k_j} c_{k_j} \implies \omega^{k_j} = 1.$$

Therefore, h must divide each k_j. If d divides each k_j with $d > h$, then $\gamma^{-k_j} = 1$ for $\gamma = e^{2\pi i/d}$. Hence $\gamma\lambda \in \sigma(\mathbf{A})$ for each $\lambda \in \sigma(\mathbf{A})$ because $c(\gamma\lambda) = 0$. But this means that $\sigma(\mathbf{A})$ is invariant under a rotation through an angle $(2\pi/d) < (2\pi/h)$, which, by (8.3.15), is impossible. ■

Example 8.3.5

Problem: Find the index of imprimitivity of $\mathbf{A} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 0 \end{pmatrix}$.

Solution: Using the principal minors to compute the characteristic equation as illustrated in Example 7.1.2 (p. 496) produces the characteristic equation

$$c(x) = x^4 - 5x^2 + 4 = 0,$$

so that $k_1 = 2$ and $k_2 = 4$. Since $\gcd\{2, 4\} = 2$, it follows that $h = 2$. The characteristic equation is relatively simple in this example, so the eigenvalues can be explicitly determined to be $\{\pm 2, \pm 1\}$. This corroborates the fact that $h = 2$. Notice also that this illustrates the property that $\sigma(\mathbf{A})$ is invariant under rotation through an angle $2\pi/h = \pi$.

More is known about nonnegative matrices than what has been presented here—in fact, there are entire books on the subject. But before moving on to applications, there is a result that Frobenius discovered in 1912 that is worth mentioning because it completely reveals the structure of an imprimitive matrix.

Frobenius Form

For each imprimitive matrix $\mathbf{A}$ with index of imprimitivity $h > 1$, there exists a permutation matrix $\mathbf{P}$ such that

$$\mathbf{P}^T\mathbf{A}\mathbf{P} = \begin{pmatrix} \mathbf{0} & \mathbf{A}_{12} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{A}_{23} & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{A}_{h-1,h} \\ \mathbf{A}_{h1} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \end{pmatrix},$$

where the zero blocks on the main diagonal are square.

Example 8.3.6

Leontief's[91] Input–Output Economic Model. Suppose that n major industries in a closed economic system each make one commodity, and let a *J-unit* be what industry J produces that sells for \$1. For example, the Boeing Company makes airplanes, and the Champion Company makes rivets, so a *BOEING-unit* is only a tiny fraction of an airplane, but a *CHAMPION-unit* might be several rivets. If

$$0 \le s_j = \#\ \textit{J-units} \text{ produced by industry } J \text{ each year, and if}$$
$$0 \le a_{ij} = \#\ \textit{I-units} \text{ needed to produce one } \textit{J-unit},$$

then

$$a_{ij}s_j = \#\ \textit{I-units} \text{ consumed by industry } J \text{ each year, and}$$
$$\sum_{j=1}^{n} a_{ij}s_j = \#\ \textit{I-units} \text{ consumed by all industries each year,}$$

so

$$d_i = s_i - \sum_{j=1}^{n} a_{ij}s_j = \#\ \textit{I-units} \text{ available to the public (nonindustry) each year.}$$

Consider $\mathbf{d} = (d_1, d_2, \ldots, d_n)^T$ to be the public ***demand vector,*** and think of $\mathbf{s} = (s_1, s_2, \ldots, s_n)^T$ as the industrial ***supply vector.***

Problem: Determine the supply $\mathbf{s} \ge \mathbf{0}$ that is required to satisfy a given demand $\mathbf{d} \ge \mathbf{0}$.

Solution: At first glance the problem seems to be trivial because the equations $d_i = s_i - \sum_{j=1}^{n} a_{ij}s_j$ translate to $(\mathbf{I} - \mathbf{A})\mathbf{s} = \mathbf{d}$, so if $\mathbf{I} - \mathbf{A}$ is nonsingular, then $\mathbf{s} = (\mathbf{I} - \mathbf{A})^{-1}\mathbf{d}$. The catch is that this solution may have negative components in spite of the fact that $\mathbf{A} \ge \mathbf{0}$. So something must be added. It's not unreasonable to assume that major industries are *strongly connected* in the sense that the commodity of each industry is either directly or indirectly needed to produce all commodities in the system. In other words, it's reasonable to assume that

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91 Wassily Leontief (1906–1999) was the 1973 Nobel Laureate in Economics. He was born in St. Petersburg (now Leningrad), where his father was a professor of economics. After receiving his undergraduate degree in economics at the University of Leningrad in 1925, Leontief went to the University of Berlin to earn a Ph.D. degree. He migrated to New York in 1931 and moved to Harvard University in 1932, where he became Professor of Economics in 1946. Leontief spent a significant portion of his career developing and applying his input–output analysis, which eventually led to the famous "Leontief paradox." In the U.S. economy of the 1950s, labor was considered to be scarce while capital was presumed to be abundant, so the prevailing thought was that U.S. foreign trade was predicated on trading capital-intensive goods for labor-intensive goods. But Leontief's input–output tables revealed that just the opposite was true, and this contributed to his fame. One of Leontief's secret weapons was the computer. He made use of large-scale computing techniques (relative to the technology of the 1940s and 1950s), and he was among the first to put the Mark I (one of the first electronic computers) to work on nonmilitary projects in 1943.

$\mathcal{G}(\mathbf{A})$ is a strongly connected graph so that in addition to being nonnegative, $\mathbf{A}$ is an *irreducible* matrix. Furthermore, it's not unreasonable to assume that $\rho(\mathbf{A}) < 1$. To understand why, notice that the j^{th} column sum of $\mathbf{A}$ is

$$c_j = \sum_{i=1}^{n} a_{ij} = \text{total number of all units required to make one } J\text{-unit}$$
$$= \text{total number of dollars spent by } J \text{ to create \$1 of revenue.}$$

In a healthy economy all major industries should have $c_j \le 1$, and there should be at least one major industry such that $c_j < 1$. This means that there exists a matrix $\mathbf{E} \ge \mathbf{0}$, but $\mathbf{E} \ne \mathbf{0}$, such that each column sum of $\mathbf{A} + \mathbf{E}$ is 1, so

$$\mathbf{e}^T(\mathbf{A} + \mathbf{E}) = \mathbf{e}^T, \quad \text{where} \quad \mathbf{e}^T \text{ is the row of all 1's.}$$

This forces $\rho(\mathbf{A}) < 1$; otherwise the Perron vector $\mathbf{p} > \mathbf{0}$ for $\mathbf{A}$ can be used to write

$$1 = \mathbf{e}^T\mathbf{p} = \mathbf{e}^T(\mathbf{A} + \mathbf{E})\mathbf{p} = 1 + \mathbf{e}^T\mathbf{E}\mathbf{p} > 1$$

because

$$\mathbf{E} \ge \mathbf{0},\ \mathbf{E} \ne \mathbf{0},\ \mathbf{p} > \mathbf{0} \implies \mathbf{E}\mathbf{p} > \mathbf{0}.$$

(Conditions weaker than the column-sum condition can also force $\rho(\mathbf{A}) < 1$—see Example 7.10.3 on p. 620.) The assumption that $\mathbf{A}$ is a nonnegative irreducible matrix whose spectral radius is $\rho(\mathbf{A}) < 1$ combined with the Neumann series (p. 618) provides the conclusion that

$$(\mathbf{I} - \mathbf{A})^{-1} = \sum_{k=0}^{\infty} \mathbf{A}^k > \mathbf{0}.$$

Positivity is guaranteed by the irreducibility of $\mathbf{A}$ because the same argument given on p. 672 that is to prove (8.3.5) also applies here. Therefore, for each demand vector $\mathbf{d} \ge \mathbf{0}$, there exists a unique supply vector given by $\mathbf{s} = (\mathbf{I} - \mathbf{A})^{-1}\mathbf{d}$, which is necessarily positive. The fact that $(\mathbf{I} - \mathbf{A})^{-1} > \mathbf{0}$ and $\mathbf{s} > \mathbf{0}$ leads to the interesting conclusion that an increase in public demand by just *one* unit from a *single* industry will force an increase in the output of *all* industries.

Note: The matrix $\mathbf{I} - \mathbf{A}$ is an M-matrix as defined and discussed in Example 7.10.7 (p. 626). The realization that M-matrices are naturally present in economic models provided some of the motivation for studying M-matrices during the first half of the twentieth century. Some of the M-matrix properties listed on p. 626 were independently discovered and formulated in economic terms.

Example 8.3.7

Leslie Population Age Distribution Model. Divide a population of females into age groups $G_1, G_2, \ldots, G_n$, where each group covers the same number of years. For example,

$$\begin{aligned} G_1 &= \text{all females under age 10,} \\ G_2 &= \text{all females from age 10 up to 20,} \\ G_1 &= \text{all females from age 20 up to 30,} \\ &\vdots \end{aligned}$$

Consider discrete points in time, say $t = 0, 1, 2, \ldots$ years, and let b_k and s_k denote the birth rate and survival rate for females in G_k. That is, let

$$\begin{aligned} b_k &= \text{Expected number of daughters produced by a female in } G_k, \\ s_k &= \text{Proportion of females in } G_k \text{ at time } t \text{ that are in } G_{k+1} \text{ at time } t+1. \end{aligned}$$

If

$$f_k(t) = \text{Number of females in } G_k \text{ at time } t,$$

then it follows that

$$f_1(t+1) = f_1(t)b_1 + f_2(t)b_2 + \cdots + f_n(t)b_n$$

and (8.3.17)

$$f_k(t+1) = f_{k-1}(t)s_{k-1} \quad \text{for } k = 2, 3, \ldots, n.$$

Furthermore,

$$F_k(t) = \frac{f_k(t)}{f_1(t) + f_2(t) + \cdots + f_n(t)} = \% \text{ of population in } G_k \text{ at time } t.$$

The vector $\mathbf{F}(t) = (F_1(t), F_2(t), \ldots, F_n(t))^T$ represents the *population age distribution* at time t, and, provided that it exists, $\mathbf{F}^\star = \lim_{t\to\infty} \mathbf{F}(t)$ is the *long-run (or steady-state) age distribution.*

Problem: Assuming that $s_1, \ldots, s_n$ and $b_2, \ldots, b_n$ are positive, explain why the population age distribution approaches a steady state, and then describe it. In other words, show that $\mathbf{F}^\star = \lim_{t\to\infty} \mathbf{F}(t)$ exists, and determine its value.

Solution: The equations in (8.3.17) constitute a system of homogeneous difference equations that can be written in matrix form as

$$\mathbf{f}(t+1) = \mathbf{Lf}(t), \quad \text{where} \quad \mathbf{L} = \begin{pmatrix} b_1 & b_2 & \cdots & b_{n-1} & b_n \\ s_1 & 0 & \cdots & \cdots & 0 \\ 0 & s_2 & 0 & & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & s_n & 0 \end{pmatrix}_{n\times n}. \tag{8.3.18}$$

The matrix $\mathbf{L}$ is called the ***Leslie matrix*** in honor of P. H. Leslie who used this model in 1945. Notice that in addition to being nonnegative, $\mathbf{L}$ is also irreducible when $s_1, \ldots, s_n$ and $b_2, \ldots, b_n$ are positive because the graph $\mathcal{G}(\mathbf{L})$ is strongly connected. Moreover, $\mathbf{L}$ is primitive. This is obvious if in addition to $s_1, \ldots, s_n$ and $b_2, \ldots, b_n$ being positive we have $b_1 > 0$ (recall Example 8.3.3 on p. 678). But even if $b_1 = 0$, $\mathbf{L}$ is still primitive because $\mathbf{L}^{n+2} > \mathbf{0}$ (recall (8.3.16) on p. 678). The technique on p. 679 also can be used to show primitivity (Exercise 8.3.11). Consequently, (8.3.10) on p. 674 guarantees that

$$\lim_{t\to\infty} \left(\frac{\mathbf{L}}{r}\right)^t = \mathbf{G} = \frac{\mathbf{p}\mathbf{q}^T}{\mathbf{q}^T\mathbf{p}} > \mathbf{0},$$

where $\mathbf{p} > \mathbf{0}$ and $\mathbf{q} > \mathbf{0}$ are the respective Perron vectors for $\mathbf{L}$ and $\mathbf{L}^T$. If we combine this with the fact that the solution to the system of difference equations in (8.3.18) is $\mathbf{f}(t) = \mathbf{L}^t\mathbf{f}(0)$ (p. 617), and if we assume that $\mathbf{f}(0) \neq \mathbf{0}$, then we arrive at the conclusion that

$$\lim_{t\to\infty} \frac{\mathbf{f}(t)}{r^t} = \mathbf{G}\mathbf{f}(0) = \mathbf{p}\left(\frac{\mathbf{q}^T\mathbf{f}(0)}{\mathbf{q}^T\mathbf{p}}\right) \quad \text{and} \quad \lim_{t\to\infty} \left\|\frac{\mathbf{f}(t)}{r^t}\right\|_1 = \frac{\mathbf{q}^T\mathbf{f}(0)}{\mathbf{q}^T\mathbf{p}} > 0 \qquad (8.3.19)$$

(because $\|\star\|_1$ is a continuous function—Exercise 5.1.7 on p. 277). Now

$$F_k(t) = \frac{f_k(t)}{\|\mathbf{f}(t)\|_1} = \%\text{ of population that is in } G_k \text{ at time } t$$

is the quantity of interest, and (8.3.19) allows us to conclude that

$$\begin{aligned}\mathbf{F}^\star = \lim_{t\to\infty} \mathbf{F}(t) &= \lim_{t\to\infty} \frac{\mathbf{f}(t)}{\|\mathbf{f}(t)\|_1} = \lim_{t\to\infty} \frac{\mathbf{f}(t)/r^t}{\|\mathbf{f}(t)\|_1/r^t} \\ &= \frac{\lim_{t\to\infty} \mathbf{f}(t)/r^t}{\lim_{t\to\infty} \|\mathbf{f}(t)\|_1/r^t} = \mathbf{p} \quad \text{(the Perron vector!)}.\end{aligned}$$

In other words, while the numbers in the various age groups may increase or decrease, depending on the value of r (Exercise 8.3.10), the proportion of individuals in each age group becomes stable as time increases. And because the steady-state age distribution is given by the Perron vector of $\mathbf{L}$, each age group must eventually contain a positive fraction of the population.

Exercises for section 8.3

8.3.1. Let $\mathbf{A} = \begin{pmatrix} 0 & 1 & 0 \\ 3 & 0 & 3 \\ 0 & 2 & 0 \end{pmatrix}$.

(a) Show that $\mathbf{A}$ is irreducible.
(b) Find the Perron root and Perron vector for $\mathbf{A}$.
(c) Find the number of eigenvalues on the spectral circle of $\mathbf{A}$.

8.3.2. Suppose that the index of imprimitivity of a 5×5 nonnegative irreducible matrix $\mathbf{A}$ is $h = 3$. Explain why $\mathbf{A}$ must be singular with $alg\ mult_{\mathbf{A}}(0) = 2$.

8.3.3. Suppose that $\mathbf{A}$ is a nonnegative matrix that possesses a positive spectral radius and a corresponding positive eigenvector. Does this force $\mathbf{A}$ to be irreducible?

8.3.4. Without computing the eigenvalues or the characteristic polynomial, explain why $\sigma(\mathbf{P}_n) = \{1, \omega, \omega^2, \ldots, \omega^{n-1}\}$, where $\omega = e^{2\pi i/n}$ for

$$\mathbf{P}_n = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

8.3.5. Determine whether $\mathbf{A} = \begin{pmatrix} 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 7 & 0 \\ 2 & 0 & 0 & 0 & 0 \\ 0 & 9 & 2 & 0 & 4 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$ is reducible or irreducible.

8.3.6. Determine whether the matrix $\mathbf{A}$ in Exercise 8.3.5 is primitive or imprimitive.

8.3.7. A matrix $\mathbf{S}_{n \times n} \geq \mathbf{0}$ having row sums less than or equal to 1 with at least one row sum less than 1 is called a ***substochastic matrix.***

(a) Explain why $\rho(\mathbf{S}) \leq 1$ for every substochastic matrix.

(b) Prove that $\rho(\mathbf{S}) < 1$ for every *irreducible* substochastic matrix.

8.3.8. A nonnegative matrix for which each row sum is 1 is called a ***stochastic matrix*** (some say *row*-stochastic). Prove that if $\mathbf{A}_{n \times n}$ is nonnegative and irreducible with $r = \rho(\mathbf{A})$, then $\mathbf{A}$ is similar to $r\mathbf{P}$ for some irreducible stochastic matrix $\mathbf{P}$. **Hint:** Consider $\mathbf{D} = \begin{pmatrix} p_1 & 0 & \cdots & 0 \\ 0 & p_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & p_n \end{pmatrix}$, where the p_k's are the components of the Perron vector for $\mathbf{A}$.

8.3.9. Wielandt constructed the matrix $\mathbf{W}_n = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ 1 & 1 & 0 & \cdots & 0 \end{pmatrix}$ to show that $\mathbf{W}^{n^2-2n+2} > \mathbf{0}$, but $[\mathbf{W}^{n^2-2n+1}]_{11} = 0$. Verify that this is true for $n = 4$.

8.3.10. In the Leslie population model on p. 683, explain what happens to the vector $\mathbf{f}(t)$ as $t \to \infty$ depending on whether $r < 1$, $r = 1$, or $r > 1$.

8.3.11. Use the characteristic equation as described on p. 679 to show that the Leslie matrix in (8.3.18) is primitive even if $b_1 = 0$ (assuming all other b_k 's and s_k 's are positive).

8.3.12. A matrix $\mathbf{A} \in \Re^{n \times n}$ is said to be ***essentially positive*** if $\mathbf{A}$ is irreducible and $a_{ij} \geq 0$ for every $i \neq j$. Prove that each of the following statements is equivalent to saying that $\mathbf{A}$ is essentially positive.

(a) There exists some $\alpha \in \Re$ such that $\mathbf{A} + \alpha\mathbf{I}$ is primitive.
(b) $e^{t\mathbf{A}} > \mathbf{0}$ for all $t > 0$.

8.3.13. Let $\mathbf{A}$ be an essentially positive matrix as defined in Exercise 8.3.12. Prove that each of the following statements is true.

(a) $\mathbf{A}$ has an eigenpair $(\xi, \mathbf{x})$, where ξ is real and $\mathbf{x} > \mathbf{0}$.
(b) If λ is any eigenvalue for $\mathbf{A}$ other than ξ, then $\mathrm{Re}\,(\lambda) < \xi$.
(c) ξ increases when any entry in $\mathbf{A}$ is increased.

8.3.14. Let $\mathbf{A} \geq \mathbf{0}$ be an irreducible matrix, and let $a_{ij}^{(k)}$ denote entries in $\mathbf{A}^k$. Prove that $\mathbf{A}$ is primitive if and only if

$$\rho\,(\mathbf{A}) = \lim_{k \to \infty} \left[a_{ij}^{(k)} \right]^{1/k}.$$